# Prefiltering and Robustness of GPC with Structured Perturbations

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**Abstract.** This paper deals with prefiltering for improving the robust stability of Generalized Predictive Control (GPC) against structured perturbations. Some guidelines for choosing an optimal value of the prefilter $T$ have been widely accepted in the literature. However, this work shows how these guidelines are not always adequate and can even worsen robustness. Thus, once a given value of the prefilter $T$ has been chosen a deep robustness analysis is needed.

**Keywords:** Model Based Predictive Control, GPC, Prefitering, Structured Perturbations, Stability Analysis, Robustness Analysis.

## 1 Introduction

All predictive controllers share a common methodology: at each "present" instant $t$, future process outputs $y(t+k \mid t)$ are predicted for a certain time window, $k = 1, 2, \ldots, N$, using a model of the process. The optimal control law is obtained by minimizing a cost function (only the unconstrained case is considered):

$$\mathrm{J}(\Delta u,t) = E\left\{ \sum_{j=N_1}^{N_2} \gamma(j)\left[r(t+j\mid t)-y(t+j\mid t)\right]^2 + \sum_{j=1}^{N_u} \lambda(j)\left[\Delta u(t+j-1\mid t)\right]^2 \right\} \quad (1)$$

where $E\{\cdot\}$ is the expectation operator, $N_1$ is the minimum costing horizon, $N_2$ is the maximum costing horizon, $N_u$ is the control horizon, $\gamma$ is a future errors weighting sequence, and $\lambda$ is a control weighting sequence.

Generalized Predictive Control (GPC) [1, 2] is one of the most representative predictive controllers. GPC assumes a CARIMA model (a transfer function plus a colored and integrated white noise) to describe the system dynamics:

$$A(z^{-1})y(t) = B(z^{-1})u(t) + \frac{T(z^{-1})}{\Delta}\xi(t) \quad (2)$$

where $\Delta$ is the increment operator and $\xi(t)$ represents uncorrelated zero-mean white noise. $T(z^{-1})$ is a polynomial which implements a prefilter. In practice, $T$ is not considered a model parameter but a controller parameter, as its value is chosen to

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improve the closed loop robustness. In fact, prefiltering is the most popular approach to robust design in GPC.

The effects of prefiltering on robustness were initially considered by Robinson and Clarke [3], who gave some guidelines for selecting $T$. However, these results are applied only for mean-level or dead-beat GPC. Soeterboek [4] pointed out that robustness would be enhanced by choosing $T = A(1-\alpha z^{-1})$, as $\alpha$ increases. Megías *et al.* [5] showed that $\alpha$ cannot be increased unlimitedly to improve robustness because performance would deteriorate. Yoon and Clarke [6] extended these works, suggesting that a more general guideline, $T = A(1-\alpha z^{-1})^{N_1}$, improves the robust stability because it ensures the presence of a low-pass filter in the control loop that rejects the high frequency unmodeled responses. This recommendation is accepted in the literature widely and deserves some further explanations.

GPC control structure can be expressed in a classical LTI form (Fig. 1). From it, it is easy to derive the expression of the closed loop characteristic equation:

$$\delta_0(z^{-1}) = RA_0\Delta + B_0 S z^{-1} \tag{3}$$

where $A_0$ and $B_0$ represent the transfer function of the actual plant (generally different to $A$ and $B$, the model transfer function used to design the GPC controller), $R$ and $S$ are the following polynomials

$$R = \frac{T + \sum_{i=N_1}^{N_2} k_{1i} H_i}{\sum_{i=N_1}^{N_2} k_{1i} q^{-N_2+i}}, \qquad S = \frac{\sum_{i=N_1}^{N_2} k_{1i} F_i}{\sum_{i=N_1}^{N_2} k_{1i} q^{-N_2+i}} \tag{4}$$

and $H_i$ and $F_i$ are polynomials that can be derived from some diofantine equations [1].

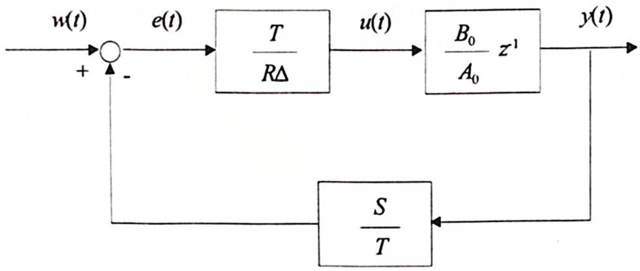


**Fig. 1.** GPC structure.

Assuming unstructured uncertainties and using the Small Gain Theorem, Yoon and Clarke [6] concluded that it is required that $S/T$ be a low-pass filter and therefore $T \neq 1$ should be used, otherwise $S/T$ $(= S)$ would be a high-pass filter. According to this, they proposed a "natural" choice for $T$:

$$T = A(1-\alpha z^{-1})^{N_1} \tag{5}$$

where $\alpha$ lies in the neighborhood of the dominant root of $A$.

The main handicap of this approach is that it has been shown that (5) is fairly simplistic. For this reason, deep analysis for each given case is needed. Sensitivity analysis (to multiplicative uncertainty, to disturbances, and to noise) has been proposed by Rossiter [7]. In practice, this analysis shows that the effect of $T$ is a trade-off between disturbances or noise rejection and robustness to model parameter uncertainty at different frequency bands, in a way that is not always beneficial. If the chosen $T$-filter does not achieve the desired sensitivities, it may not be obvious how to redesign it to improve matters.

Following this line of argument, in this paper we will study the influence of $T$ on robustness when structured perturbations are considered using tools applicable to polytopes of polynomials. It will be shown how (5) can lead to poorer robustness when these perturbations are present and how this fact can be depicted in geometrical terms.

This paper has been structured as follows: In Section 2 the GPC control of plants with structured uncertainties is introduced. Section 3 shows by examples how the general guideline for choosing $T$ is not always adequate. Finally, section 4 draws the main conclusions of this paper.

## 2 GPC with Structured Uncertainties

Structured perturbations mean that the uncertainties are in the coefficients, *i.e.* the numerator and the denominator of the actual plant are given by uncertain polynomials. Affine linear uncertainty structures will be considered. Thus, given a set of real parameters $q_i, i = 0,\ldots,l$, which can vary between a maximum and a minimum value, $q_i^- \le q_i \le q_i^+, i = 0,\ldots,l$, the coefficients of the numerator and denominator polynomials are affine linear functions of the uncertainty parameter vector $\mathbf{q} = (q_0,\ldots,q_l) \in \Re^l$, that is $a_i(\mathbf{q}) = \alpha_i^T \mathbf{q} + \beta_i, i = 0,\ldots,m$, $b_i(\mathbf{q}) = \gamma_i^T \mathbf{q} + \rho_i, i = 0,\ldots,n$ where $\alpha_i$ and $\gamma_i$ are $1 \times l$ vectors and $\beta_i$ and $\rho_i$ are scalars. Then, the actual plant is defined by the family of plants:

$$G_0(\mathbf{q},z^{-1}) = \frac{B_0(\mathbf{q},z^{-1})}{A_0(\mathbf{q},z^{-1})} = \frac{\sum_{i=0}^{n} b_i^0(\mathbf{q}) z^{-i}}{1+\sum_{i=1}^{m} a_i^0(\mathbf{q}) z^{-i}} \tag{6}$$

where ($m \ge n \ge 0$). With this structure of uncertainties, $A_0(\mathbf{q},z^{-1})$ and $B_0(\mathbf{q},z^{-1})$ *are polytopes of polynomials in* $z^{-1}$, and $G_0(\mathbf{q},z^{-1})$ *is a polytope of plants in* $z^{-1}$.

It has been shown that the family of characteristic polynomials $\delta_0(z^{-1})$ of the closed loop system constituted by a GPC controller (4) and the family of plants (6) is a polytope of polynomials [8, 9]:

$$\delta_0(z^{-1}) = R(z^{-1})A_0(\mathbf{q},z^{-1})\Delta + S(z^{-1})B_0(\mathbf{q},z^{-1})z^{-1} \tag{7}$$

This fact allows the use of a very mature theory from the point of view of the robust stability analysis. Nowadays it can be said that there are powerful results to analyze the stability and the robust performance of families of polynomials formed by interval polynomials or by polytopes of polynomials. The main tools for the analysis of polynomial families are Kharitonov Theorem [10] for interval polynomials and the Edge Theorem [11] and Rantzer Theorem [12] for polytopes of polynomials.

In order to determine which $T$ leads to the best robustness when structured uncertainties are present, we propose the analysis of the stability region in the parameter space derived from the closed loop characteristic equation. The method of Ackermann [13] will be used to draw this region because it has low computational cost and no conservatism. The stability hypersphere around the nominal process will be also analyzed.

In the following two examples will illustrate how guideline (5) influences robustness in the presence of structured uncertainties.

## 3 Examples

### 3.1 Example 1

Let us revisit an example proposed by Yoon and Clarke [6],

$$G = \frac{B}{A} = \frac{0.2}{1 - 0.8z^{-1}} \tag{8}$$

and let us assume that the plant is actually represented by the following family (interval) of plants:

$$G_0 = \frac{0.2}{1 + (-0.8 + \alpha_1)z^{-1} + \alpha_2 z^{-2}} \tag{9}$$

with $\alpha_1$ and $\alpha_2$ the uncertainty parameters. The GPC controller is tuned with the following predictive control settings: $N_1 = 1$, $N_u = 2$, $N_2 = 5$ and $\lambda = 0.01$.

**Case $T = 1$**

In the absence of prefiltering (*i.e.* $T = 1$) the polynomials $R$ and $S$ (4) are the following:

$$\begin{aligned} R(z^{-1}) &= 0.2624 \\ S(z^{-1}) &= 1.9231 - 0.9231z^{-1} \end{aligned} \tag{10}$$

and the closed loop characteristic equation is:

$$\begin{aligned}\delta_0(z,\alpha_1,\alpha_2) &= 0.2624z^3 + (-0.0877 + 0.2624\alpha_1)z^2 + \\ &+(0.0253 - 0.2624\alpha_1 + 0.2624\alpha_2)z + \\ &-0.2624\alpha_2\end{aligned} \tag{11}$$

**Case $T$=A(1− 0.8z$^{-1}$)**

Now we will follow the standard guideline (5): $T = A(1 - \alpha z^{-1})^{N_1}$, with $\alpha \in (0,1)$. It is extensively accepted that the robustness will be better when $\alpha$ lies in the neighborhood of the dominant root of $A$. Therefore, we will take $\alpha = 0.8$ as it was proposed in [6]. Thus $R$ and $S$ are

$$\begin{aligned}R(z^{-1}) &= 0.2624 - 0.0752z^{-1} + 0.0202z^{-2} \\ S(z^{-1}) &= 0.2 - 0.16z^{-1}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\delta_0(z,\alpha_1,\alpha_2) &= 0.2624z^5 + (-0.5075 + 0.2624\alpha_1)z^4 + \\ &+(0.3335 - 0.3376\alpha_1 + 0.2624\alpha_2)z^3 + \\ &+(-0.09652 + 0.0954\alpha_1 - 0.3376\alpha_2)z^2 + \\ &+(0.01616 - 0.0202\alpha_1 + 0.0954\alpha_2)z - 0.0202\alpha_2\end{aligned} \tag{13}$$

The families of polynomials (11) and (13) are polytopes as it was stated above (7). Fig. 2 shows their stability regions.

The area of the stability region for $T = 1$ is smaller than the one that follows the recommendation (5). Therefore, it could be concluded that the robust stability, in terms of the associated stability areas, has been improved with prefiltering.

However, the radius of the stability hypersphere (in this case just a circle) around the nominal process is smaller with prefiltering. Even though the stability region is bigger, from this point of view prefiltering following (5) deteriorates the robustness of the closed loop system.

### 3.2 Example 2

Now one of the examples proposed by Rossiter [7] is considered,

$$G = \frac{1}{1 - 1.4z^{-1} + 0.45z^{-2}} \tag{14}$$

assuming that the actual perturbations are structured and therefore the actual plant is given by a family (interval) of plants

$$G_0 = \frac{1}{1 + (-1.40 + \alpha_1)z^{-1} + (0.45 + \alpha_2)z^{-2}} \tag{15}$$

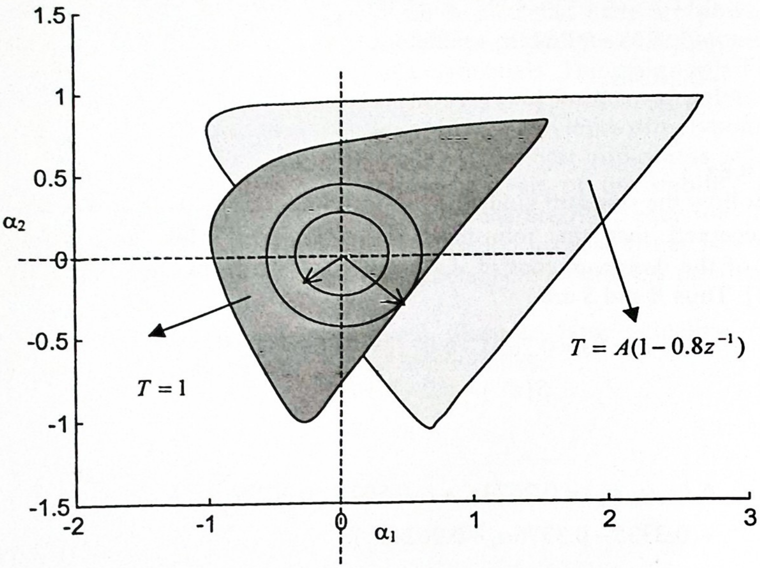


**Fig. 2.** Example 1. Stability regions for $T = 1$ and $T = A\ (1 - 0.8z^{-1})$.

with $\alpha_1$ and $\alpha_2$ the uncertainty parameters. The GPC controller has been tuned with $N_1 = 1, N_u = 1, N_2 = 10$ and $\lambda = 0.01$.

This example illustrates the situations considered in [7]: No prefiltering (*i.e.* $T = 1$, the situation rejected by Yoon and Clarke [6]) and $T$ following the general recommendation $(1-0.8z^{-1})^n$, $n = 1$ or 2. (This choice of $\alpha$ is intuitive in that if sampling at about 1/10 of the rise time, a common guideline for predictive control, then a typical dominant process pole would be around 0.8. Hence this is a sensible pole for a low-pass filter on output measurements.)

For the nominal system, the sensitivity analysis performed in [7] concludes that the inclusion of a $T$-filter has given good reductions in every sensitive function over the high frequency: the sensitivity function to multiplicative uncertainty is actually better over the whole frequency range, the output sensitivity is worse at mid and low frequencies and better at high frequencies, and the input sensitivity is better over the whole frequency range.

Now let us consider the effect of $T$-filter on structured uncertainties.

**Case $T = 1$**

The controller is given by the following expressions

$$\begin{gathered} R(z^{-1}) = 8.9245 \\ S(z^{-1}) = 9.9977 - 13.0137z^{-1} + 4.0159z^{-2} \end{gathered} \tag{16}$$

and the closed loop characteristic equation is

$$\begin{aligned}&\delta_0(z,\alpha_1,\alpha_2)=8.9245z^3+\\&+(-11.421+8.9245\alpha_1)z^2+\\&+(3.4963-8.9245\alpha_1+8.9245\alpha_2)z+\\&-8.9246\alpha_2-0.000125\end{aligned} \tag{17}$$

**Case $T=(1-0.8z^{-1})$**
Now we obtain

$$\begin{aligned}R(z^{-1})&=8.9245-0.0001z^{-1}\\S(z^{-1})&=2.8583-3.8772z^{-1}+1.2189z^{-2}\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\delta_0(z,\alpha_1,\alpha_2)=8.9245z^4+\\&+(-18.561+8.9245\alpha_1)z^3+\\&+(12.633-8.9246\alpha_1+8.9245\alpha_2)z^2+\\&+(-2.7973-0.0001\alpha_1-8.9246\alpha_2)z+\\&-0.0001\alpha_2+0.000045\end{aligned} \tag{19}$$

**Case $T=(1-0.8z^{-1})^2$**

$$\begin{aligned}R(z^{-1})&=8.9245-4.9728z^{-1}+0.0001z^{-2}\\S(z^{-1})&=0.6914-0.9632z^{-1}+0.3118z^{-2}\end{aligned} \tag{20}$$

$$\begin{aligned}&\delta_0(z,\alpha_1,\alpha_2)=8.9245z^5+(-25.7+8.9245\alpha_1)z^4+\\&+(27.482+13.897\alpha_1+8.9245\alpha_2)z^3+\\&+(-12.904+4.9729\alpha_1-0.9632\alpha_2)z^2+\\&+(2.2379-0.0001\alpha_1+4.9729\alpha_2)z+\\&-0.0001\alpha_2-0.000045\end{aligned} \tag{21}$$

Fig. 3 shows the stability regions of the polytopes (17), (19), and (21). The stability region for $T$=1 is smaller than the ones that follow the recommendation (5). However, as in the previous example the stability hypersphere around (0, 0) –absence of uncertainties– is bigger when there is no prefiltering ($T$ = 1). Therefore, in this sense robustness is not improved by the guideline.

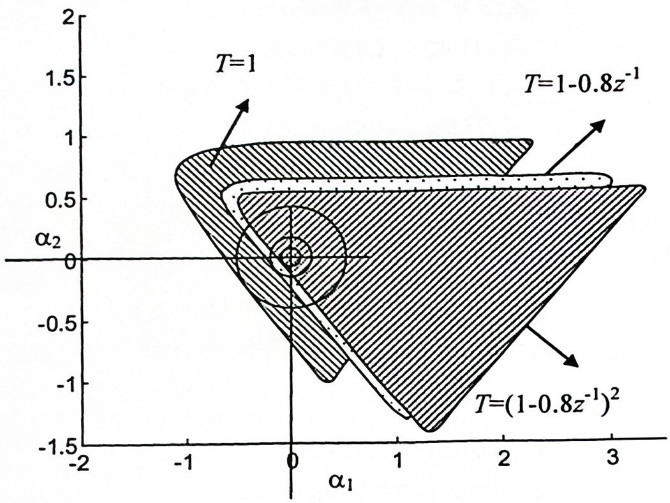


**Fig. 3.** Example 2. Stability regions for $T = 1$, $T = A\ (1 - 0.8z^{-1})$,and $T = A\ (1 - 0.8z^{-1})^2$.

## 4 Conclusions

This paper has focused on the study of the influence of prefilter $T$ on the robustness of GPC against structured uncertainties. Given that the closed loop is a polytope of polynomials, it is possible to analyze the robust stability with tools based on polytopes with ease and no conservatism.

Some examples have shown that the widely accepted guideline for choosing $T$ does not guarantee better robustness. In fact, this has been shown with simple geometrical measurements, such as the area of the stability region or the radius of the stability hypersphere in the uncertainties space.

Examples have shown that there exist "directions" in the uncertainties space where the robust stability margins are better and "directions" where they are worse. This situation is somehow similar to the sensitivity analysis when unstructured disturbances are considered, where there could be frequency bands where sensitivity is improved and bands where it is worsened.

For these reasons, no matter the type of disturbances that are present, once a given value of the prefilter $T$ has been chosen a deep analysis of robustness (sensitivity functions, stability regions, etc.) is needed.

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